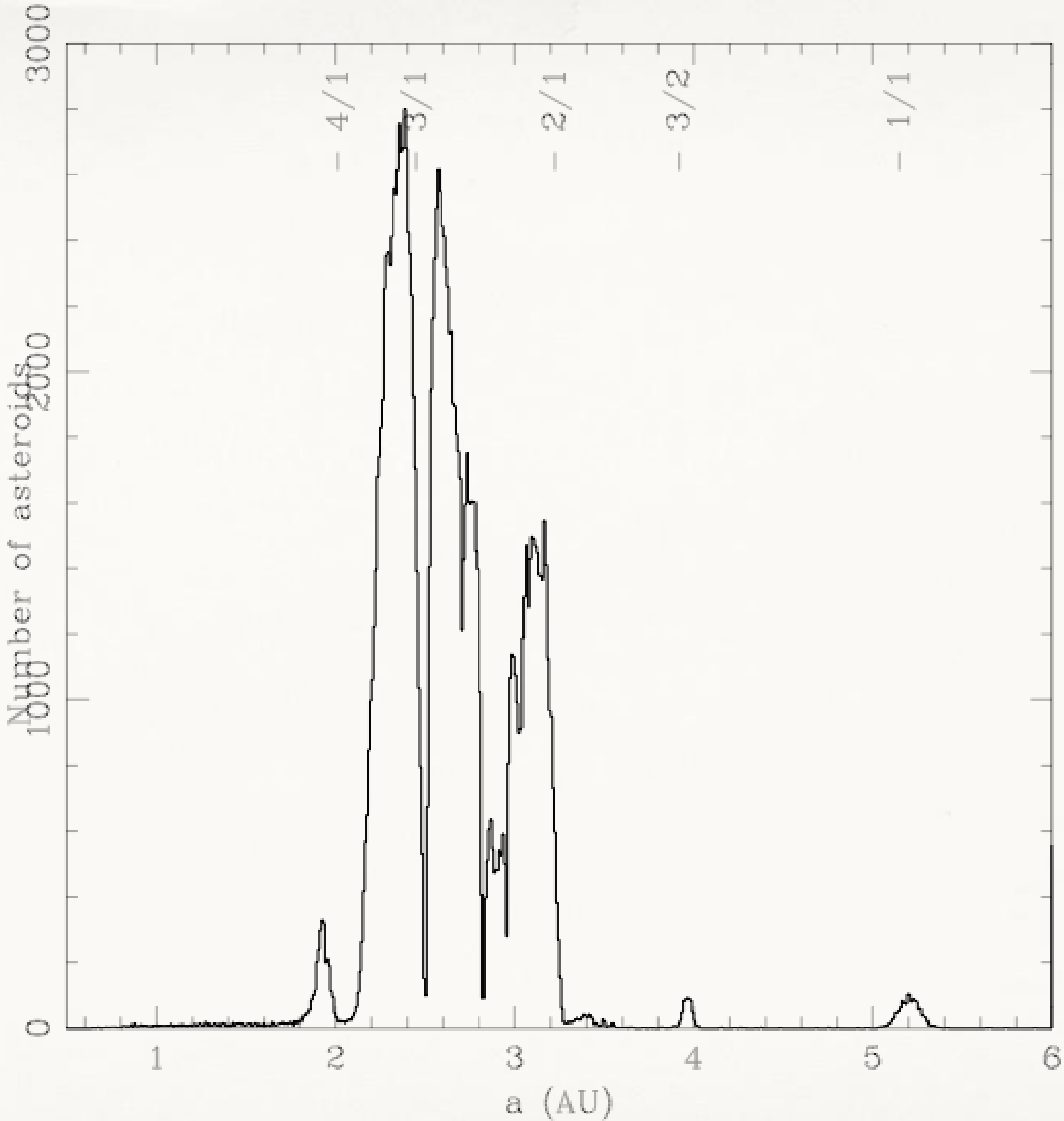
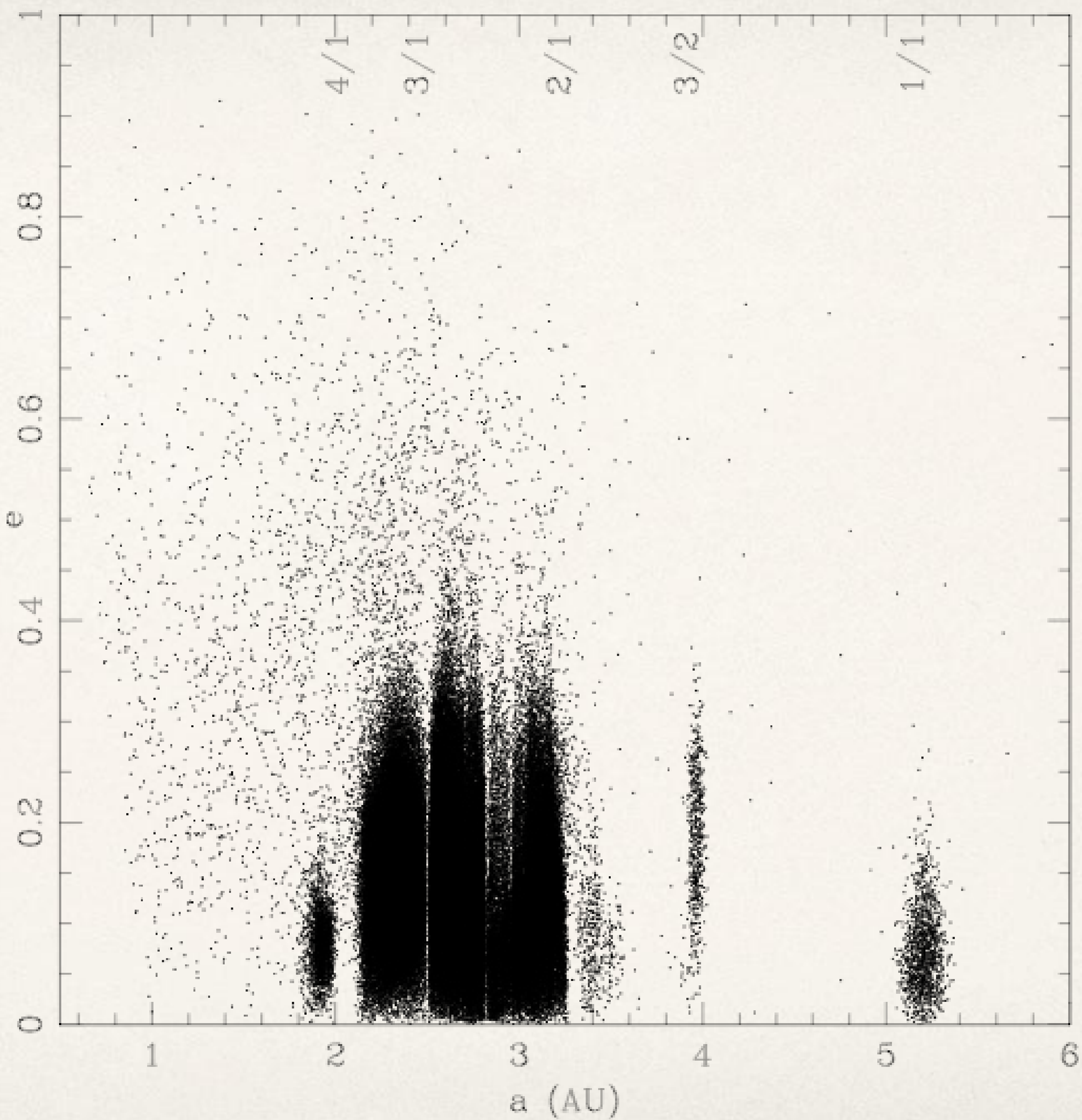
Planetary Chaos

- Phenomenology
- The Two Body Problem
- The Three Body Problem
- Resonances and Chaos
- The Four Body Problem
- Stability of Planetary Systems



The Asteroid Belt



4/1
3/1
2/1
3/2
1/1
e
0
0.2
0.4
0.6
0.8
1
1
2
3
4
5
6
a (AU)

I. THE TWO BODY PROBLEM

The motion of a body of mass m_i at position $\mathbf{x}_i$ is given by

$$m_i \frac{d^2\mathbf{x}_i}{dt^2} = -\sum_{j\neq i} \frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}(\mathbf{x}_i - \mathbf{x}_j). \tag{1.1}$$

Writing $\mathbf{r} \equiv \mathbf{x}_2 - \mathbf{x}_1$ along with $\mu \equiv G(m_1 + m_2)$, I find

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{\mu \mathbf{r}}{r^3}. \tag{1.2}$$

Taking the cross product with $\mathbf{r}$ I find $\mathbf{r} \times \ddot{\mathbf{r}} = 0$; integrating this gives $\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h}$, where $\mathbf{h}$ is a constant, or

$$r^2\dot{\theta} = h \tag{1.3}$$

Returning to equation (1.2) and extracting the radial component, one finds

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}. \tag{1.4}$$

Substituting $u = 1/r$, and using $\dot{\theta} = h/r^2$ to change the independent variable to θ results in the equation for a harmonic oscillator

$$\frac{d^2u}{d\theta^2} + u = \mu/h^2, \tag{1.5}$$

with the general solution $u = (\mu/h^2)[1 + e\cos(\theta - \varpi)]$. The quantities e and ϖ ("curly pi") are the two constants of integration. In terms of r,

$$r = \frac{h^2/\mu}{1 + e\cos(\theta - \varpi)}. \tag{1.6}$$

The conservation of angular momentum implies that

$$\frac{d\theta}{dt} = \frac{\sqrt{\mu a(1-e^2)}}{r^2} \approx n, \tag{1.7}$$

where $n = \sqrt{\mu/a^3}$. Astronomers use $\lambda \equiv nt + \varpi$, where

$$\theta(t) = \lambda + 2e\sin(\lambda - \varpi) + O\left(e^2\sin(2\lambda - 2\varpi)\right) + \dots \tag{1.8}$$

I. THE THREE BODY PROBLEM

If there are two planets orbiting a star, the equation of motion of planet 1 in the gravitational fields of the sun and planet 2 is

$$\frac{d^2\mathbf{r}_1}{dt^2} + G(M_\odot + m_1)\frac{\mathbf{r}_1}{r_1^3} = m_2 \nabla_1 R_{1,2}, \tag{1.1}$$

where

$$R_{1,2} = G\left[\frac{1}{r_{1,2}} - \frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{r_2^3}\right]. \tag{1.2}$$

The idea is then to expand the disturbing function in terms of the orbital elements of the two planetary bodies. As a first step we expand the direct term in the disturbing function in the cosine of the difference $\theta_1 - \theta_2$. The inverse distance between the two bodies is

$$\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \frac{1}{r_2}\left[1 - 2\frac{r_1}{r_2}\cos(\theta_1 - \theta_2) + \left(\frac{r_1}{r_2}\right)^2\right]^{1/2}. \tag{1.3}$$

Expanding,

$$\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \frac{1}{r_2}\sum_{l=0}^{\infty}\left(\frac{r_1}{r_2}\right)^l P_l\left(\cos(\theta_1 - \theta_2)\right). \tag{1.4}$$

The quantities $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$ and so forth are Legendre polynomials. Consider the $l = 1$ term. To first order in the eccentricities, ignoring the inclinations,

$$\begin{aligned}
\frac{r_1}{r_2^2}\cos(\theta_1 - \theta_2) &\approx \frac{a_1}{a_2^2}\left[1 - e_1\cos(\lambda_1 - \varpi_1) + 2e_2\cos(\lambda_2 - \varpi_2)\right] \\
&\times \cos\left[\lambda_1 + 2e_1\sin(\lambda_1 - \varpi_1) - \lambda_2 - 2e_2\sin(\lambda_2 - \varpi_2)\right] \\
&= \frac{a_1}{a_2^2}[\cos(\lambda_1 - \lambda_2) + e_1\cos(2\lambda_1 - \lambda_2 - \varpi_1) \\
&-\frac{3}{2}e_1\cos(\lambda_2 - \varpi_1) + 2e_2\cos(\lambda_1 - 2\lambda_2 + \varpi_2)].
\end{aligned} \tag{1.5}$$

In general, the potential contains terms of the form

$$\phi = \frac{Gm_2}{a_1}\Sigma_{\mathbf{j}}\Phi_{\mathbf{j}} e_1^{|j_3|} e_2^{|j_4|}\cos[j_1\lambda_1 + j_2\lambda_2 + j_3\varpi_1 + j_4\varpi_4]. \tag{1.6}$$

The argument of the cosine is referred to as a resonant argument.

I. RESONANCES

Roughly speaking, a resonance occurs when one of the resonant arguments is constant. A resonance can alter a and e dramatically, despite that fact that the potential is so small (of order $m_2/M_\odot \approx 1/1000$). Laplace noticed this two hundred years ago.

At the 3/1 resonance, $\lambda_1 = 3\lambda_2 \approx$ const., or $n_1 \approx 3n_2$. If we expand a_1 around this resonant value $a_{res} \approx a_2/3^{2/3}$, and define $\theta \equiv \lambda - 3\lambda_J + 2\varpi$, the motion is described by the following Hamiltonian:

$$H = \frac{\beta}{2a^2}P^2 + 2\gamma n \frac{m_J}{M_\odot}\mathcal{G} + \frac{Gm_J}{a}\Big[\Phi_1 e^2 \cos\theta + \Phi_2 e e_J \cos(\theta + \varpi_J - \varpi) + \Phi_3 e_J^2 \cos(\theta + 2\varpi_J - 2\varpi)\Big], \tag{1.1}$$

where $P \sim (a - a_{res})$ and $\mathcal{G} \sim e^2$ is the action conjugate to ϖ. The term proportional to $\mathcal{G}$ describes the precession frequency of the inner body's orbit,

$$\dot{\varpi} \equiv \frac{d\varpi}{dt} = 2\gamma \frac{m_J}{M_\odot} n \tag{1.2}$$

If $e_J = 0$, our Hamiltonian reduces to the pendulum Hamiltonian $H = P^2/(2m) + A\cos\theta$. On the separatrix of the pendulum

$$\frac{\Delta a}{a} \sim \frac{e}{\beta}\frac{m_J}{M_\odot} \tag{1.3}$$

The libration frequency is

$$\omega_0 \sim \sqrt{\beta e^2 \frac{m_J}{M_\odot}}\, n << n \tag{1.4}$$

If $e_J \neq 0$, then there are three resonances, separated by

$$\frac{\delta a}{a} \sim (\dot{\varpi} - \dot{\varpi}_J)/n \sim \frac{m_J}{M_\odot}. \tag{1.5}$$

These three resonances overlap in the asteroid belt.

Hamiltonian Dynamics

$$H = H(P, \theta, G, \omega)$$

$$\frac{dP}{dt} = -\frac{\partial H}{\partial \theta}, \quad \frac{dG}{dt} = -\frac{\partial H}{\partial \omega}$$

$$\frac{d\theta}{dt} = \frac{\partial H}{\partial P}, \quad \frac{d\omega}{dt} = \frac{\partial H}{\partial G}$$

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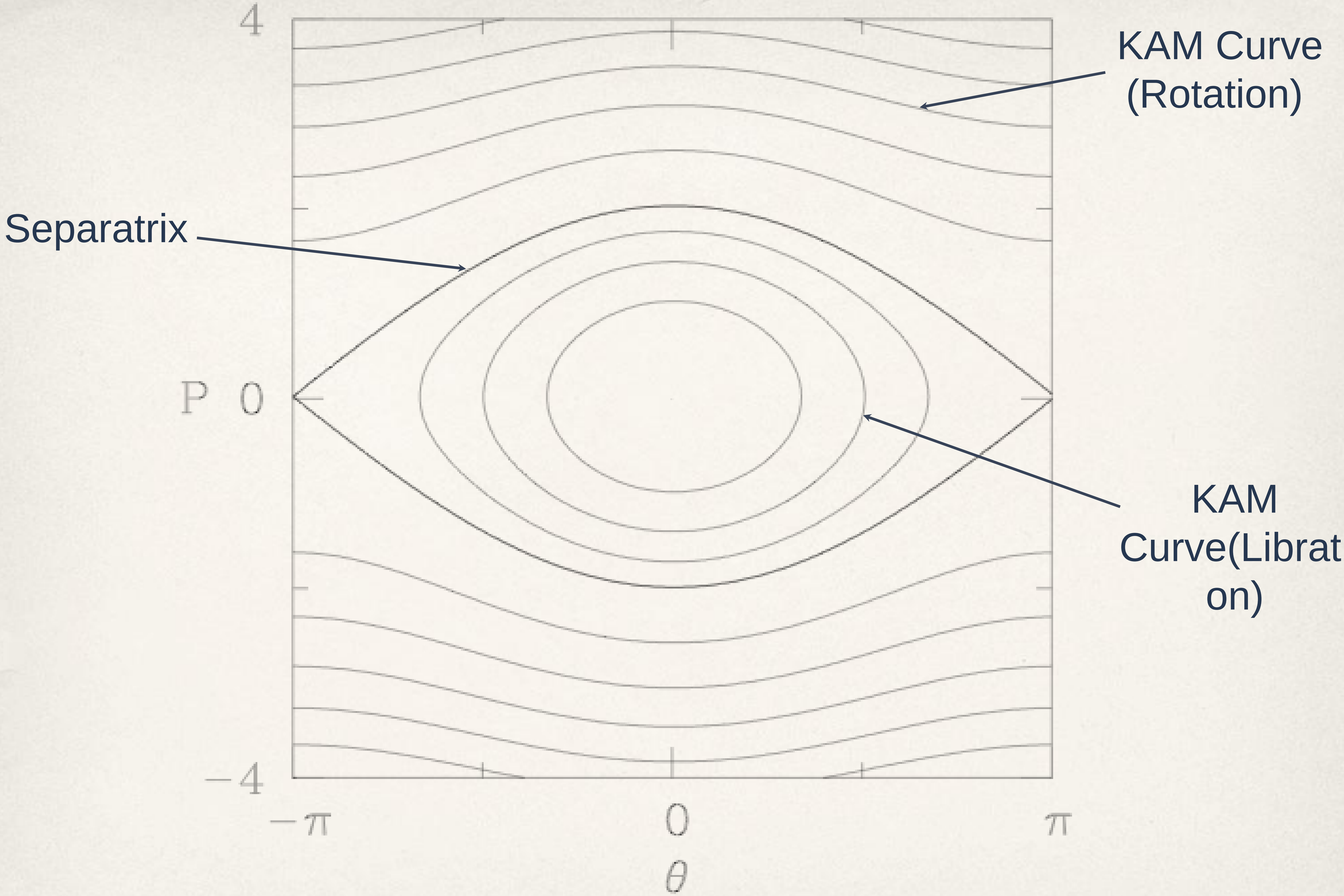

The Pendulum Phase Space

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where $P \sim (a - a_{res})$ and $\mathcal{G} \sim e^2$ is the action conjugate to ϖ. The term proportional to $\mathcal{G}$ describes the precession frequency of the inner body's orbit,

$$\dot{\varpi} \equiv \frac{d\varpi}{dt} = 2\gamma \frac{m_J}{M_\odot} n \quad (1.2)$$

If $e_J = 0$, our Hamiltonian reduces to the pendulum Hamiltonian $H = P^2/(2m) + A\cos\theta$. On the separatrix of the pendulum

$$\frac{\Delta a}{a} \sim \frac{e}{\beta}\frac{m_J}{M_\odot} \quad (1.3)$$

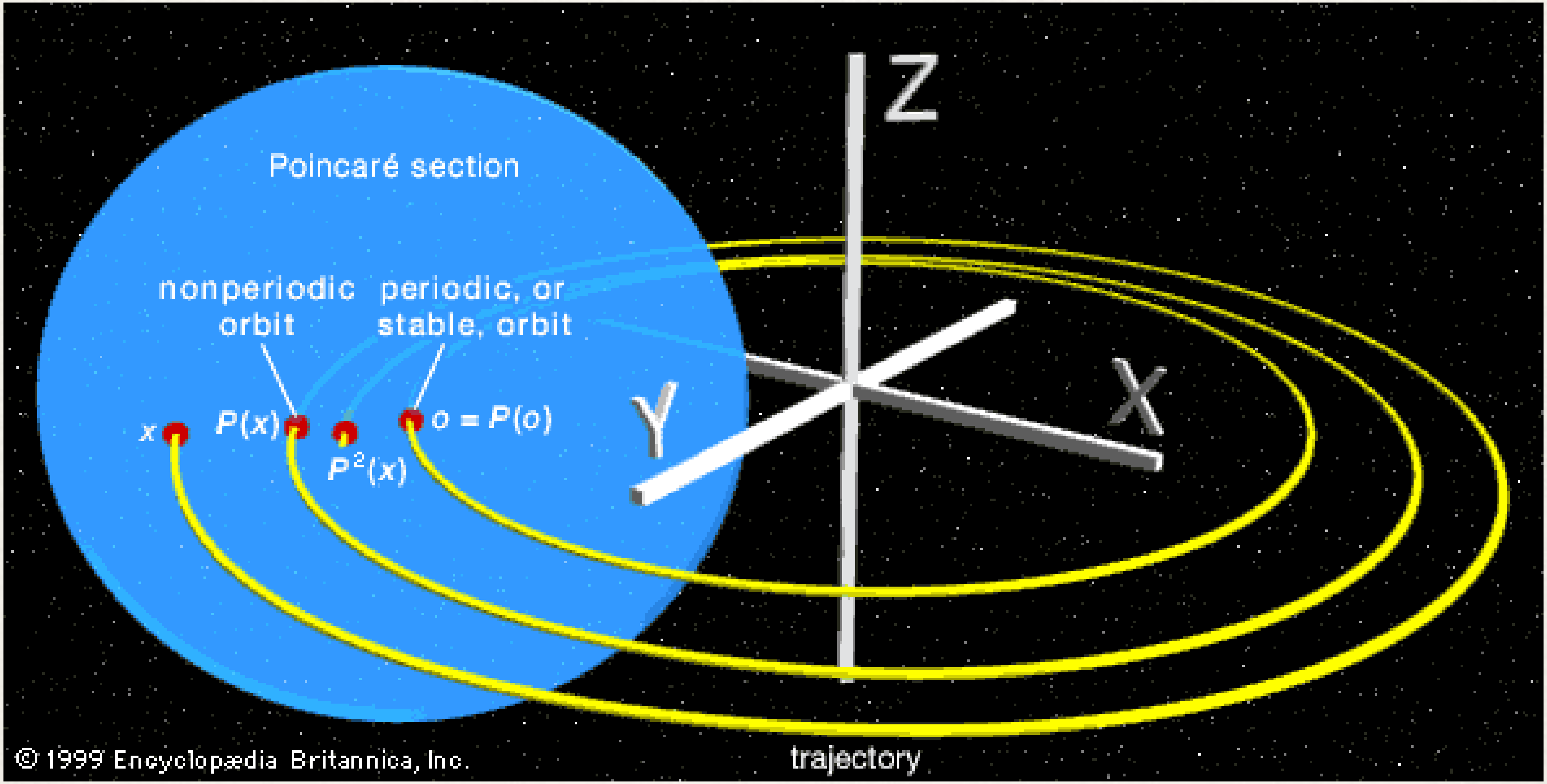
The libration frequency is

$$\omega_0 \sim \sqrt{\beta e^2 \frac{m_J}{M_\odot}}\, n << n \quad (1.4)$$

If $e_J \neq 0$, then there are three resonances, separated by

$$\frac{\delta a}{a} \sim (\dot{\varpi} - \dot{\varpi}_J)/n \sim \frac{m_J}{M_\odot}. \quad (1.5)$$

These three resonances overlap in the asteroid belt.


Poincaré section
nonperiodic orbit
periodic, or stable, orbit
x
P(x)
P2(x)
o = P(o)
Z
Y
X
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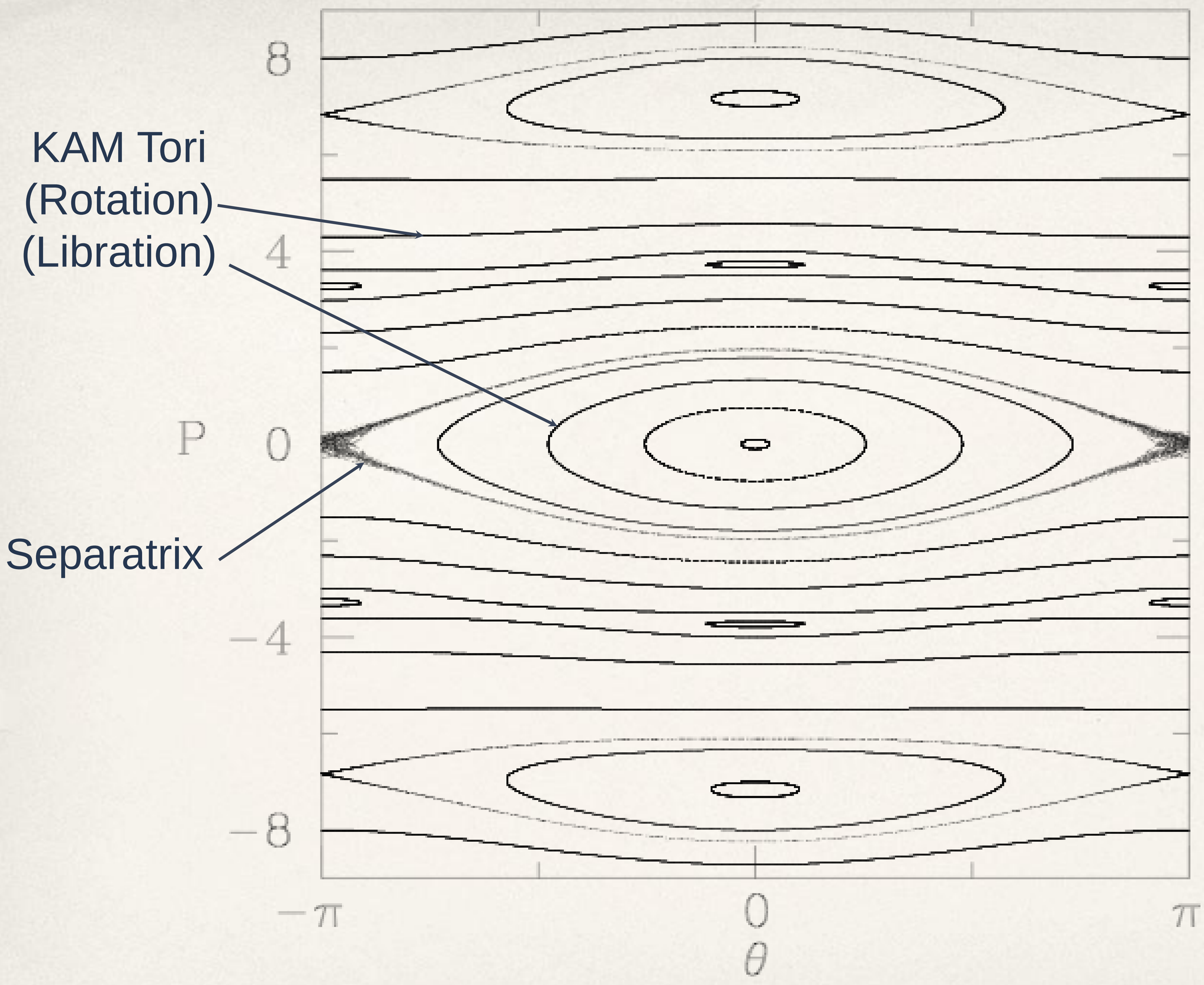
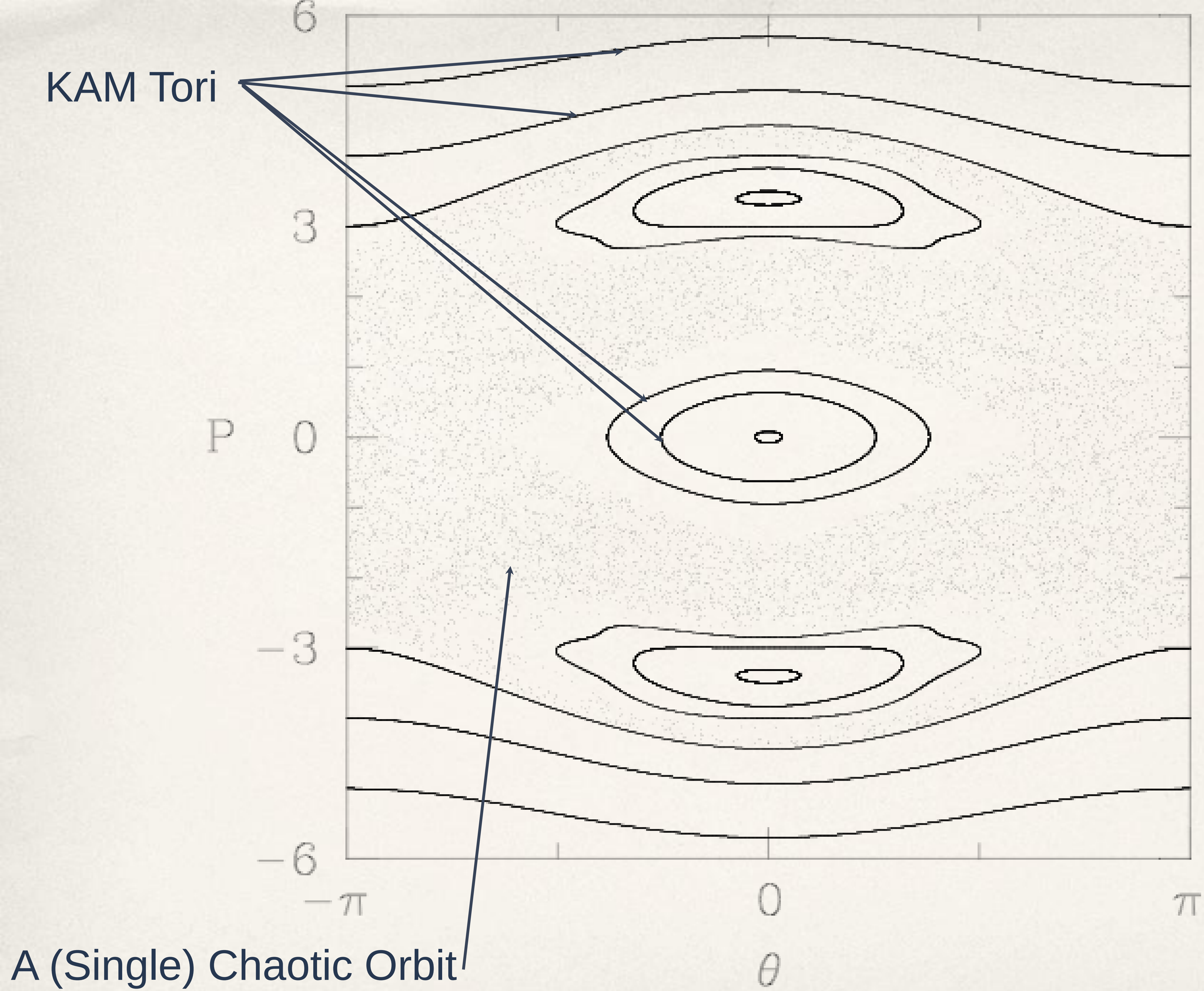
trajectory



Well Separated Resonances

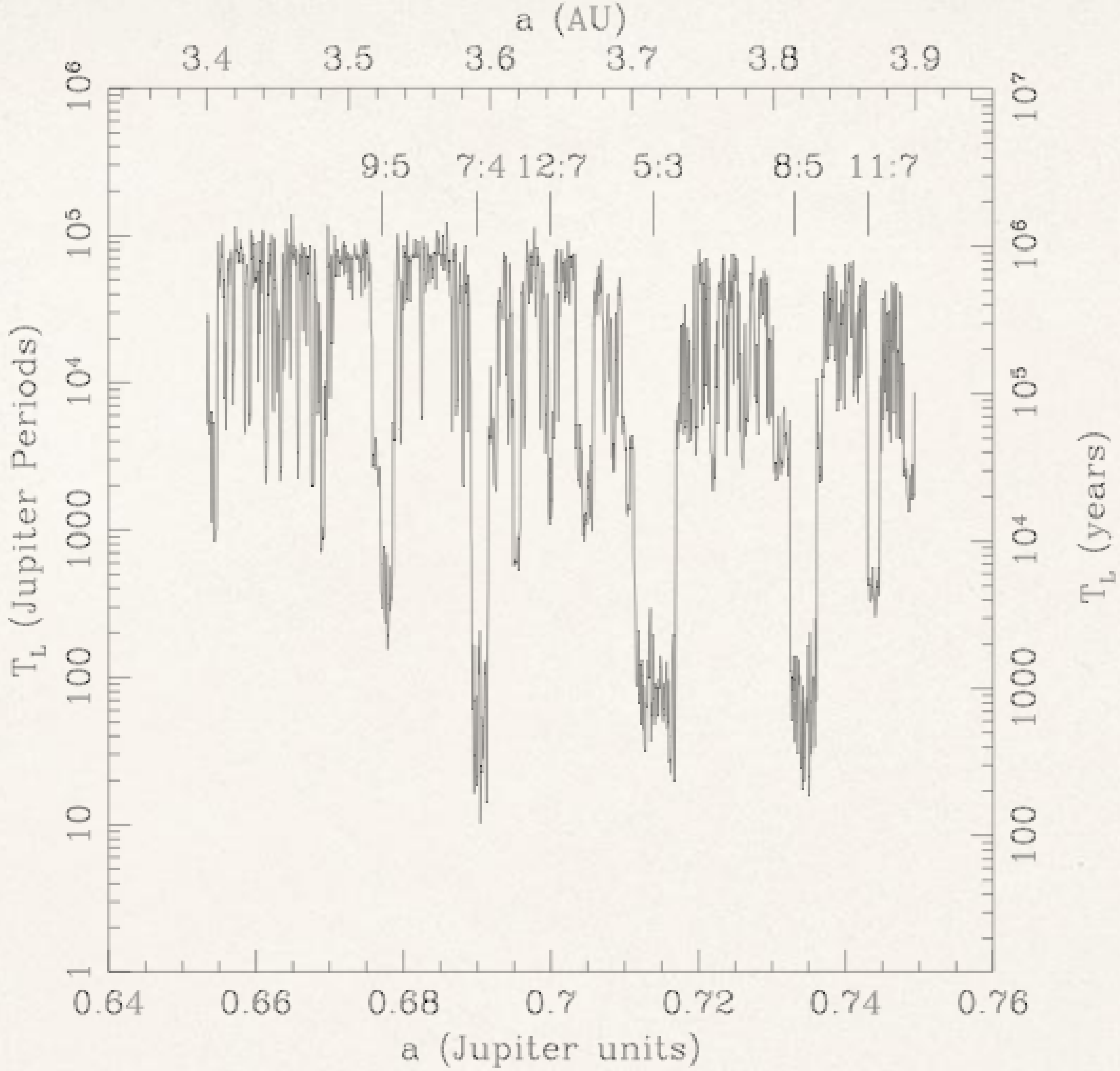


Resonance Width Equal to Separation

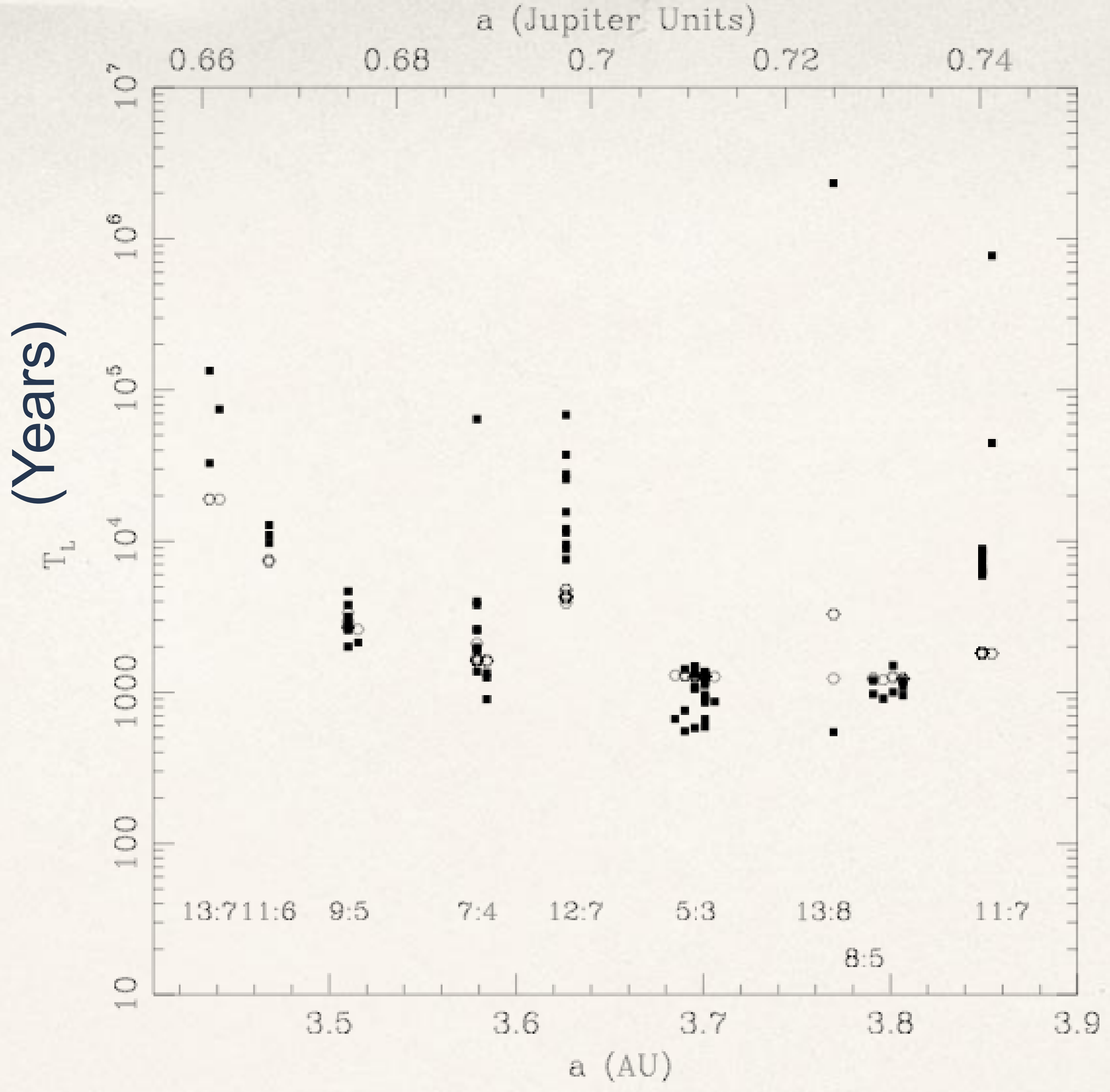
Characterizing Chaos

1. Lyapunov Times

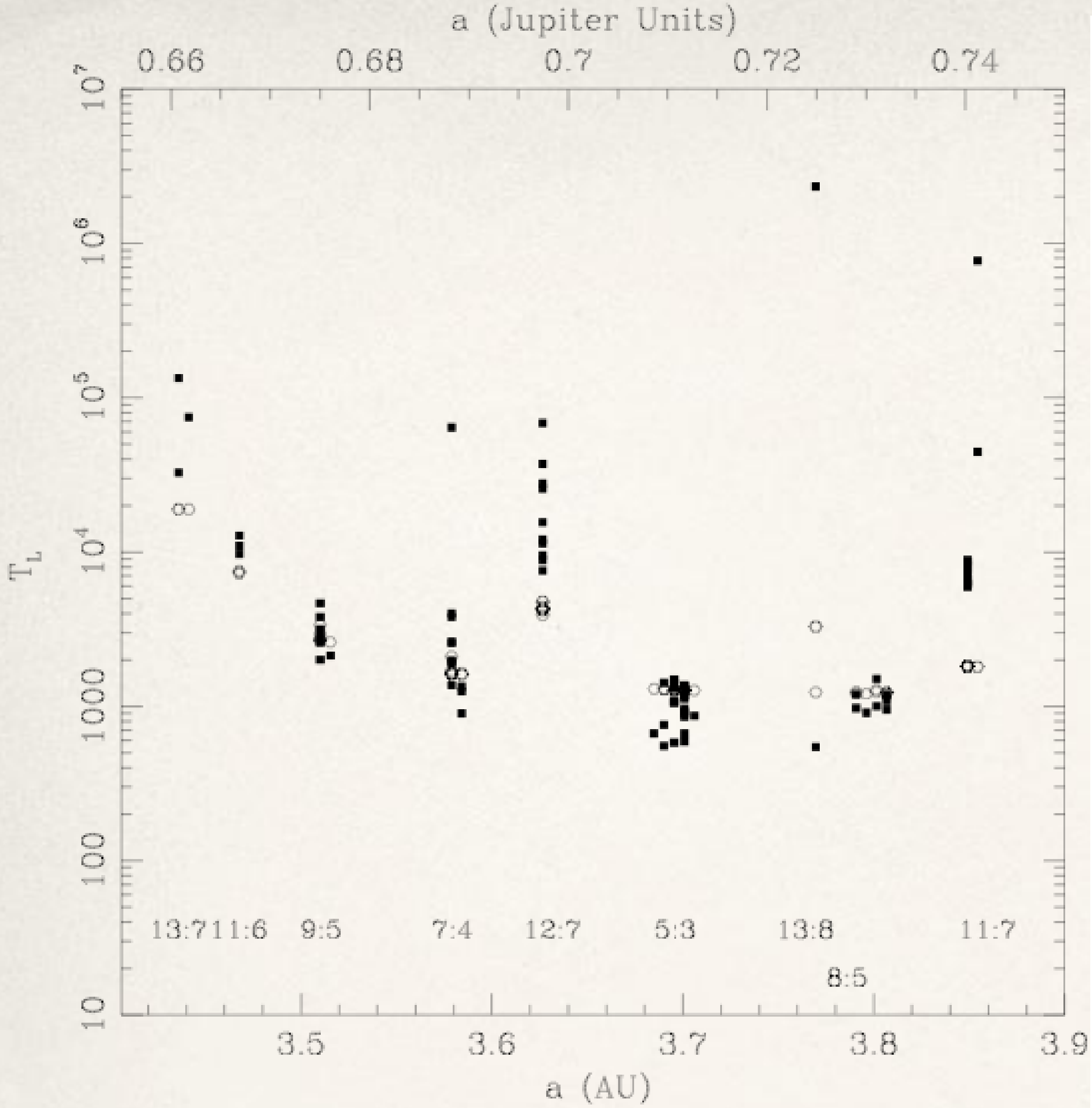
In order for two (or more) resonances to overlap, their widths must be comparable to their separation. In other words, the precession period must be comparable to the libration period of the resonance. The time for nearby initial conditions to separate, called the Lyapunov time, is then given by that same quantity.



a (AU)
3.4
3.5
3.6
3.7
3.8
3.9
9:5
7:4
12:7
5:3
8:5
11:7
T_L (Jupiter Periods)
T_L (years)
a (Jupiter units)
0.64
0.66
0.68
0.7
0.72
0.74
0.76



Calculated Lyapunov Times in the Asteroid Belt



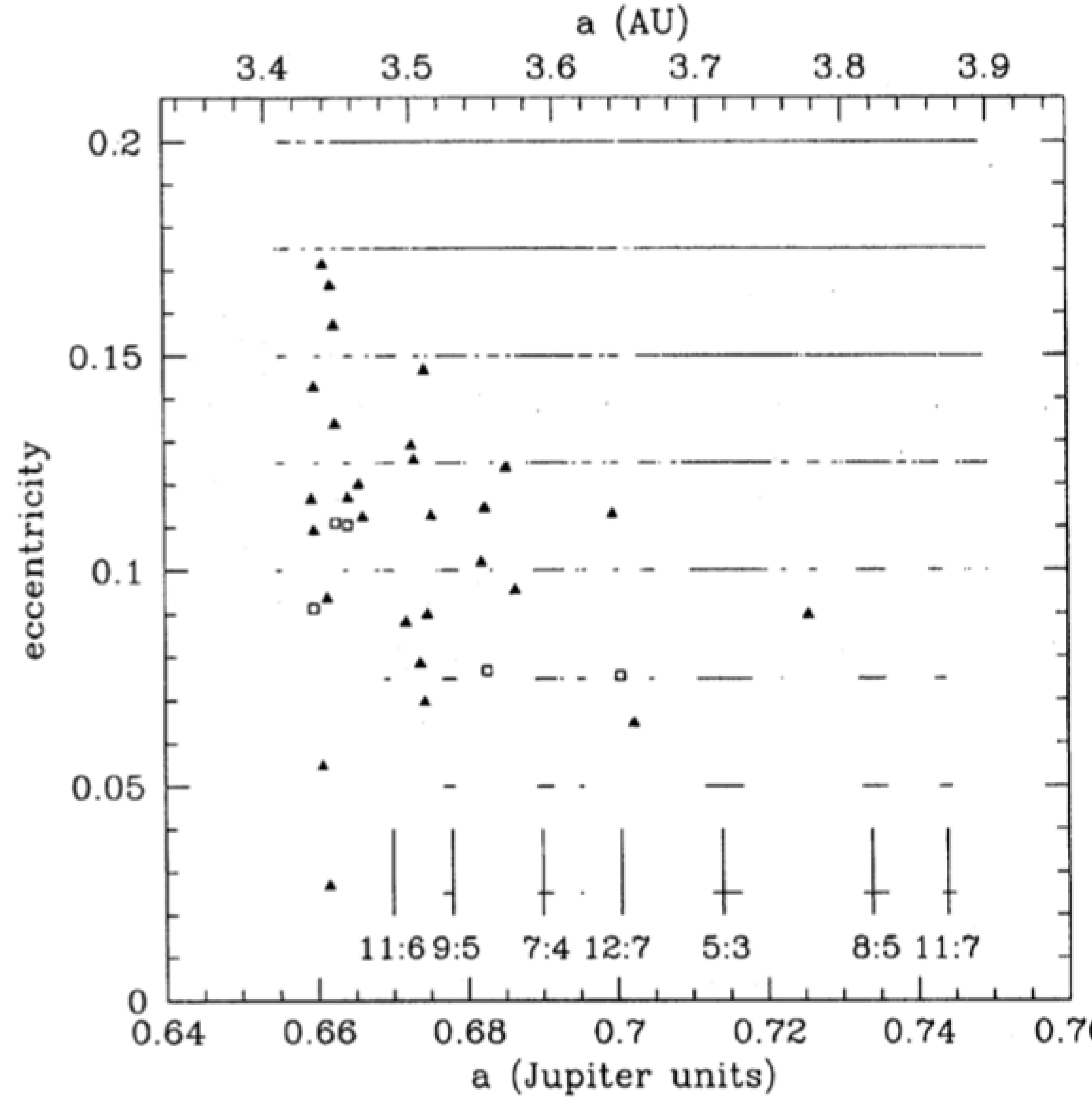
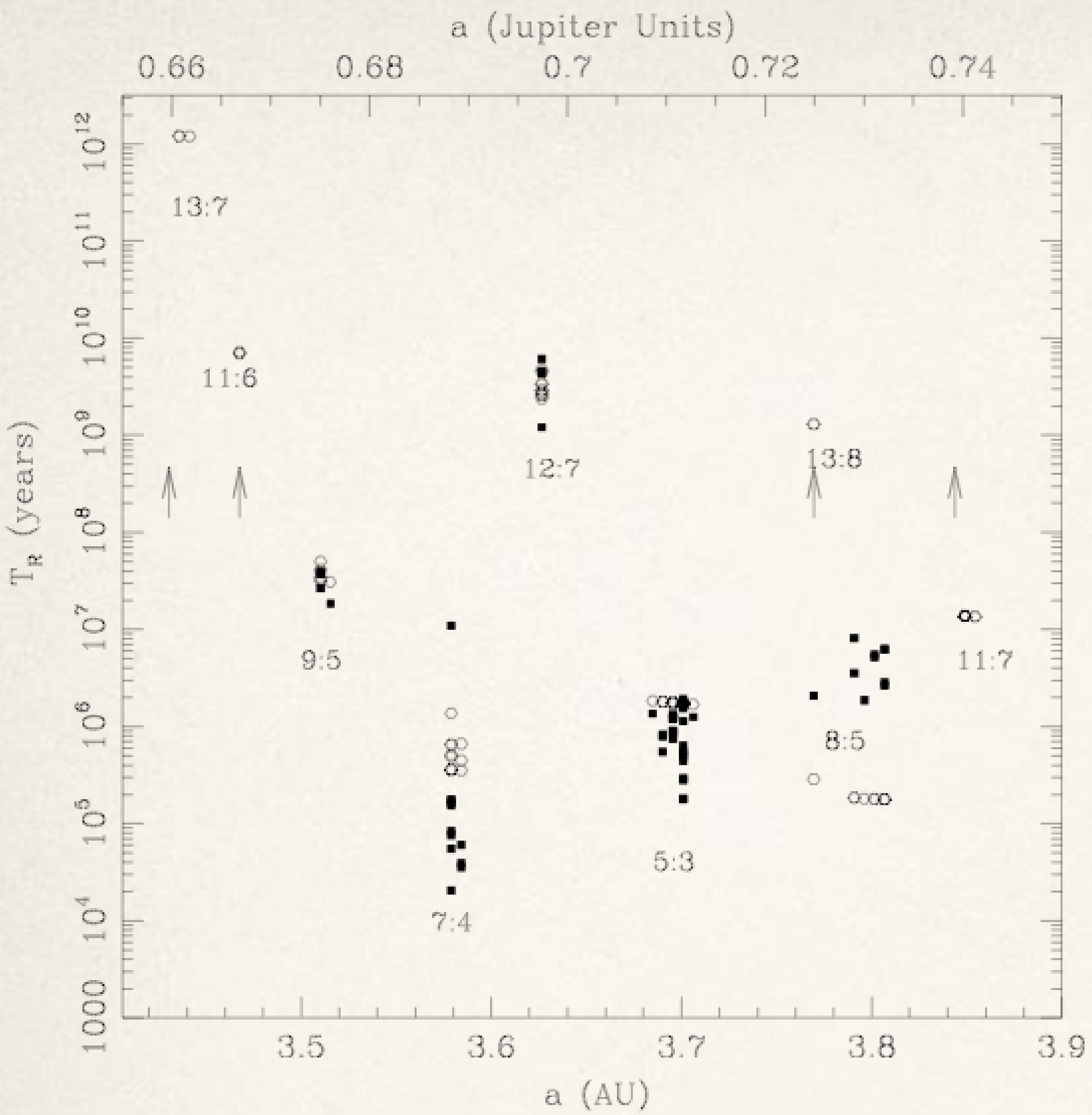


FIG. 7. For each set of initial conditions a point is plotted when $T_L < 10^{3.86}$ years. The outlines in $a-e$ space of the chaotic zones associated with mean motion resonances are evident. The real outer belt asteroids, integrated to initial longitudes consistent with the test particle initial conditions, are plotted as squares and triangles. The squares represent those asteroids confirmed to be in high-order mean motion resonances. Although most of the outer belt asteroids have chaotic trajectories according to Table 2, they lie outside of the chaotic zones with the shortest Lyapunov times.

Characterizing Chaos

2. Removal Times

In the Asteroid belt, objects that are injected into a resonance (perhaps by a collision between two asteroids, or by the Yarkovsky effect) undergo a random walk in both a and e. Since e has a reflecting lower boundary (e=0), and an absorbing upper boundary (when e is large enough that the asteroid either crosses the orbit of Jupiter, Mars, or Earth, or enters a secular resonance and hits the sun), the time to remove the asteroid is the time to diffuse from the initial (small) e to large e.



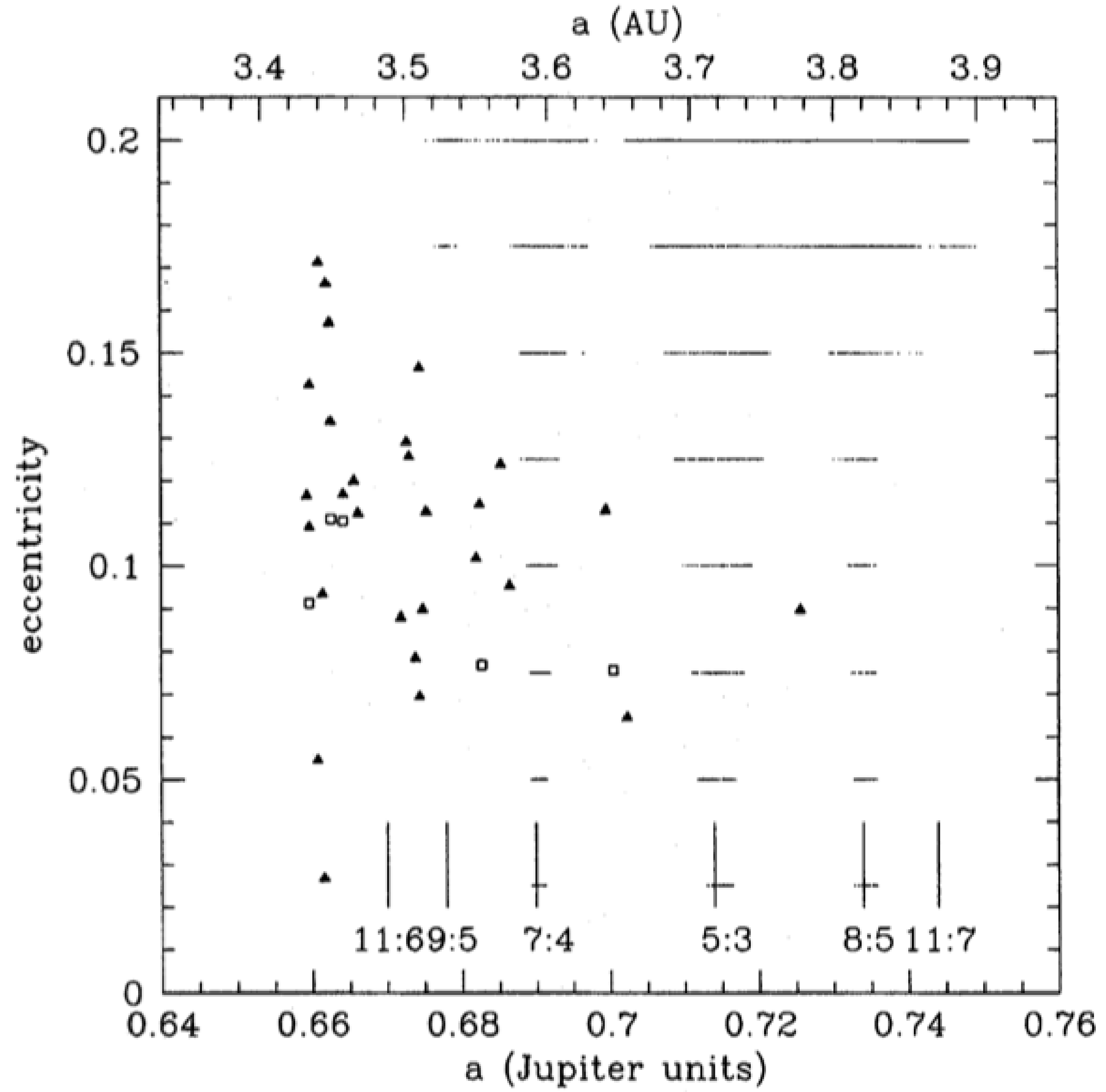


FIG. 8. For each set of initial conditions a point is plotted if the test particle underwent a close encounter with Jupiter before the end of the integration. Nearly all test particles in the 7:4, 5:3, and 8:5 were removed, regardless of initial eccentricity. At high eccentricities test particles were removed from the 9:5 resonance as well. The regions from which test particles are removed correspond to the chaotic zones seen in the previous figure. On the time scale of these integrations no test particles were removed from the low-eccentricity regions of the 9:5 resonances or from any part of the 11:6 resonance, although no asteroids are currently seen in those regions.

Calculated Removal Times vs. Observed Asteroids

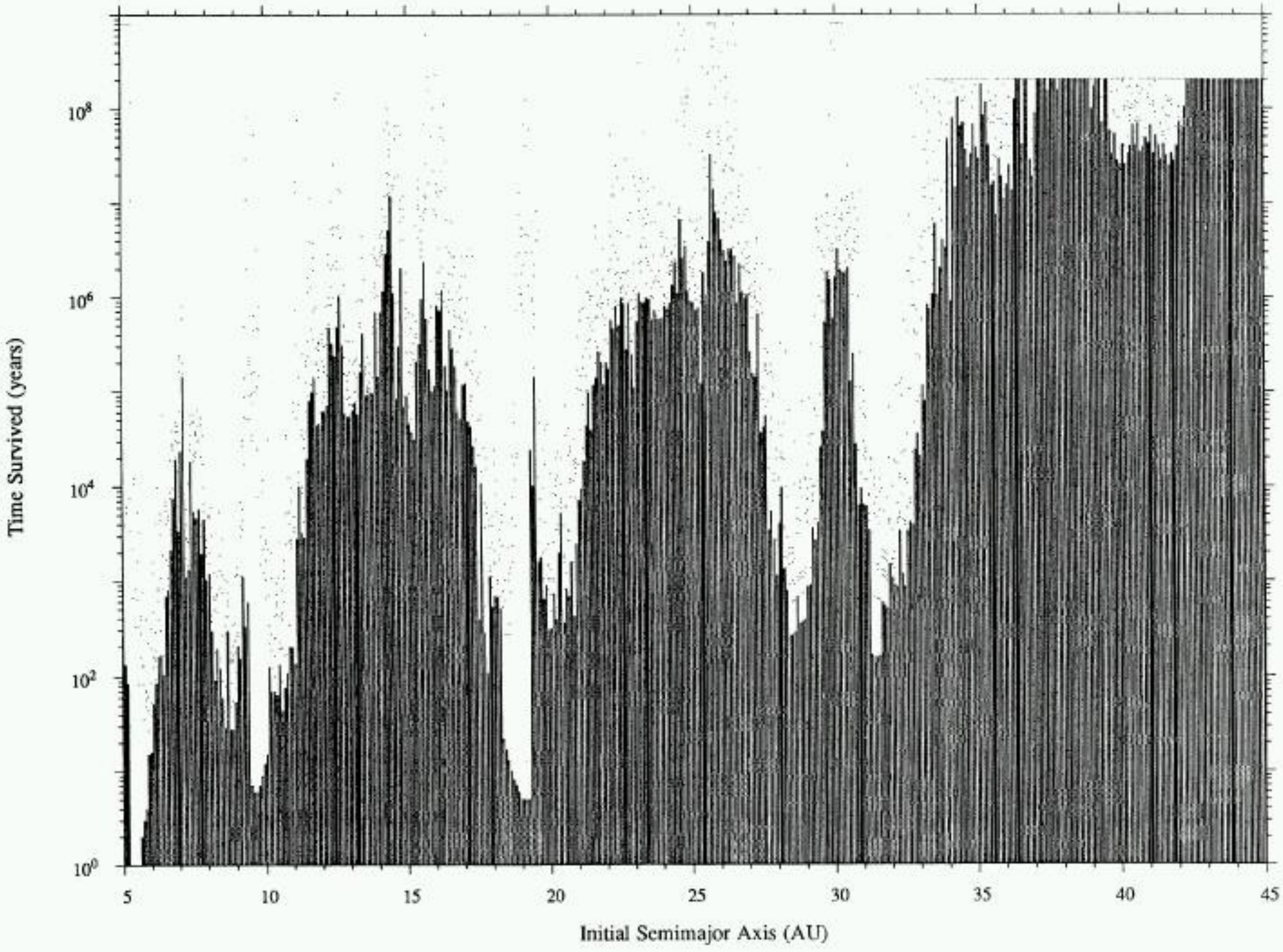


FIG. 2. The time survived by each test particle is plotted as a function of initial semimajor axis. For each semimajor axis bin, six test particles were started at different longitudes. The vertical bars mark the minimum of the six termination times. The points mark the termination time of the other five test particles. The scatter of points gives an idea of the spread of termination times for any given semimajor axis. The envelope at the top is the mark of those test particles surviving the full integration. The spikes at 5.2, 9.5, 19.2, and 30.1 AU, at the semimajor axes of the planets, correspond to test particles librating in Trojan or horseshoelike orbits before close encounter. Interior to Neptune the integration extends to 800 million yr; exterior to Neptune the integration reaches 200 million yr. Beyond about 43 AU all the test particles survive the full integration.

I. THE FOUR BODY PROBLEM

The dynamics of a planetary system become increasingly rich as more planets are added. For example, consider the three planets Jupiter, Saturn, and Uranus. Jupiter and Saturn are near a 5/2 resonance, as first noted by Laplace. This resonance forces changes in the mean longitude of both planets, an effect known as the great inequality. It also forces changes in the eccentricity of both planets. For example, Saturn induces changes in the eccentricity of Jupiter:

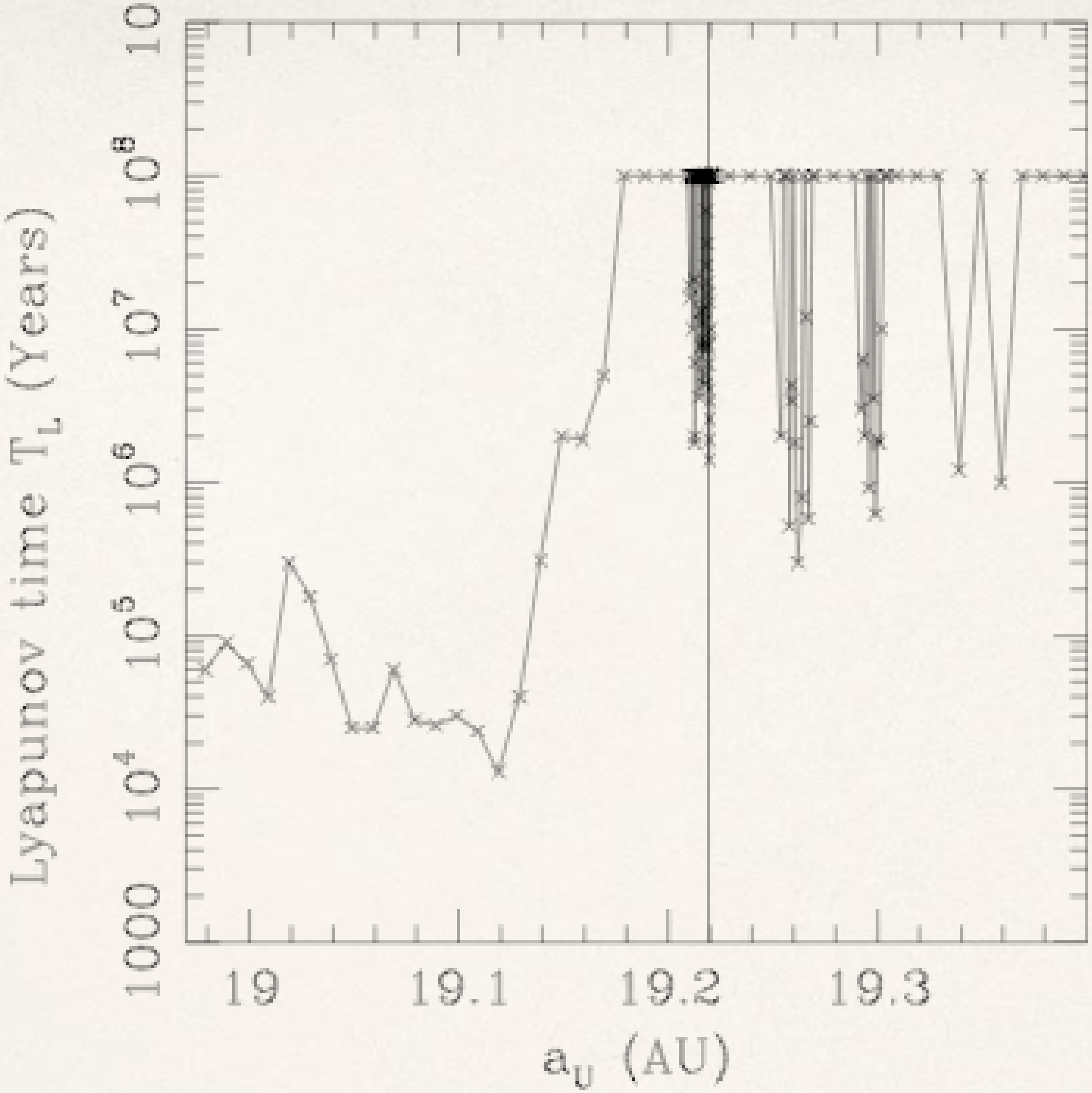
$$\Delta e_J \sin \varpi_J \approx \frac{m_S}{M_\odot (2 - 5n_S/n_J)} \frac{a_J}{a_S} \sum_{p>0} \Phi^{(2,5)}_{k,p,q,r} e_S^k e_J^{p-1} i_J^q i_S^r \times \sin[2\lambda_J - 5\lambda_S + k\varpi_S + (p-1)\varpi_J + q\Omega_J + r\Omega_S]. \tag{1.1}$$

Jupiter also perturbs Uranus. Since Uranus has a mass of 1/20 M_J, we can consider Uranus to be an asteroid, and use the expression we obtained previously. If we do so, we find that Uranus is near, but not in, a 7/1 mean motion resonance with Jupiter.

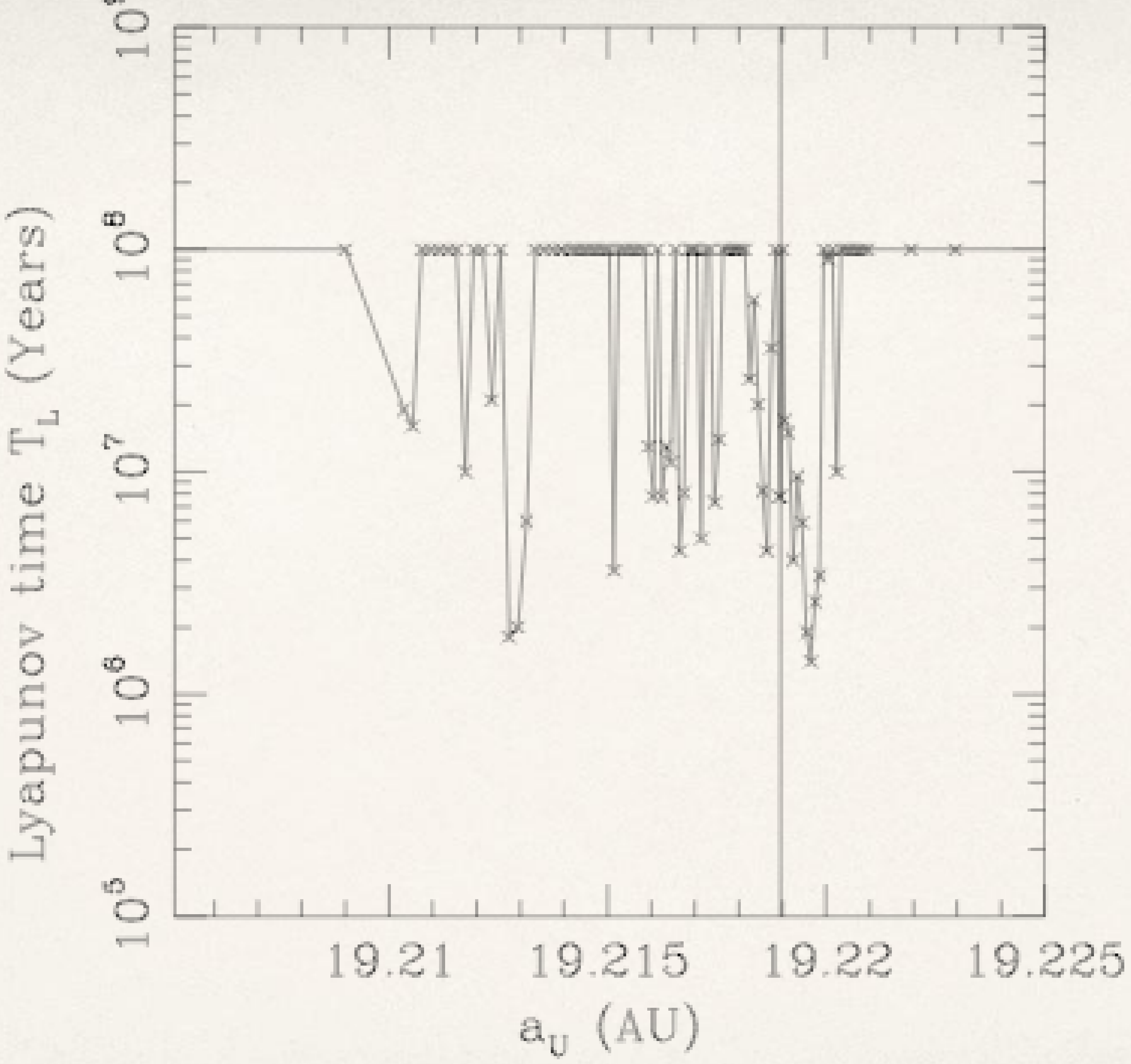
However, if we now account for the fact that Jupiter's orbit is perturbed by Saturn, we find that the potential contains a term

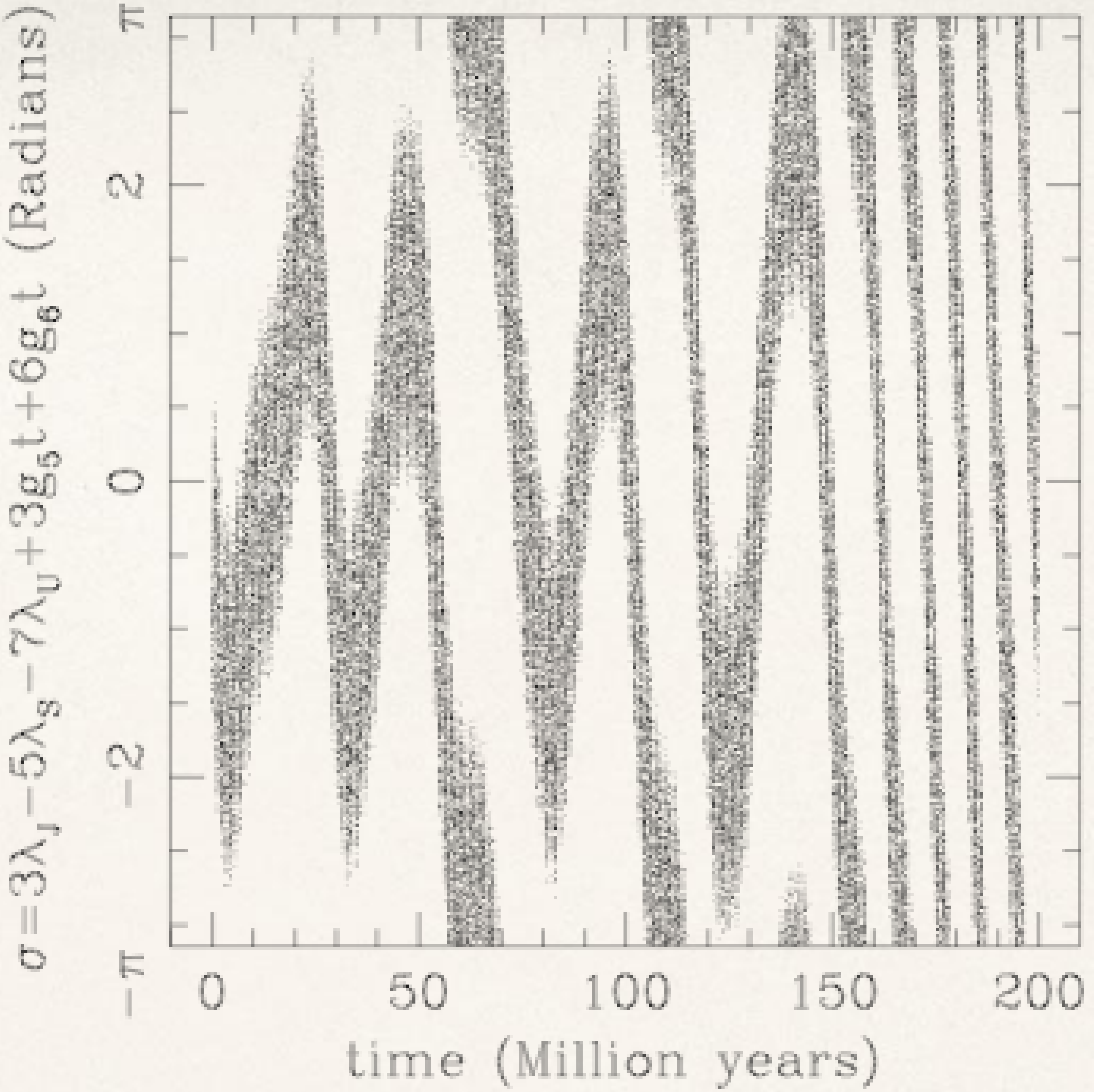
$$\Delta V \approx -\frac{GM_J}{a_U} \frac{m_S}{M_\odot} \epsilon \frac{a_S}{a_J} \sum_{p=0}^{5} (6-p) \Phi^{(7,1)}_{6-p,p,0,0} \phi^{(5,2)}_{2,1,0,0} e_J^{5-p} e_U^p e_S^2 \times \sin[3\lambda_J - 5\lambda_S - 7\lambda_U + 7\varpi_J + p(\varpi_U - \varpi_J) + 2\varpi_S]. \tag{1.2}$$

There are six mean motion resonances in this expression. Using the correct values for the planetary masses and orbital elements, these resonances just barely overlap, and produce chaotic motion amongst the giant planets.



Lyapunov time T_L (Years)
1000
10^4
10^5
10^6
10^7
10^8
10^9
19
19.1
19.2
19.3
a_U (AU)


Lyapunov time T_L (Years)
10^5
10^6
10^7
10^8
10^9
19.21
19.215
19.22
19.225
a_U (AU)



$\sigma=3\lambda_J-5\lambda_S-7\lambda_U+3g_5t+6g_6t$ (Radians)
π
2
0
−2
−π
0
50
100
150
200
time (Million years)